Theorems on Lorentz Space

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ABSTRACT: In this paper we introduce a generalization of the Lorentz space .And prove some theorems about it.

المستخلص : قدمنا في هذا البحث توسيع لفضاء لورنز مع بعض المبرهنات حول هذا التوسيع .

1-Introduction :

Let f be a complex-valued measurable function defined on a σ- finite measure space $(X, \mathcal{A}, \mu)$. For $s \geq 0$,define μf the distribution function of f as

$$\mu f(s) = \mu\{x \in X: |f(x)| > s\}.$$ [**Arora , Datt and Verma , 2007**]

By f^*we mean the non –increasing rearrangement of f given as

$$f^*(t) = inf\{s > 0: \mu f(s) \leq t\}, \qquad t \geq 0\,.$$

For $t > 0$, let

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*\,(s)\,ds\,.$$

For a measurable function f on X, define

$$\|f\|_{pq} = \left\{\frac{q}{p}\int_0^\infty \left(t^{1/p} f^{**}(t)\right)^q \frac{dt}{t}\right\} \; 0 < q, p < 1$$

The Lorentz space $L(p, q)$ consists of those complex – valued measurable functions f on X such that $\|f\|_{pq} < \infty$. For more on Lorentz space one can refer to [**Bennet and Sharpley1988, Hunt 1966, Lorentz 1950, Stein and Weiss,1971**].

Let $\mathrm{T}: X \rightarrow X$ be a measurable ($\mathrm{T}^{-1}(E) \in \mathcal{A}$,for $E \in \mathcal{A}$) non-singular transformation $\left(\mu\left(\mathrm{T}^{-1}(E)\right) = 0 \; whenever \; \mu(E) = 0\right)$ and u a complex –valued measurable function defined on X.

We define a linear transformation $\mathcal{W} = \mathcal{W}_{u,\mathrm{T}}$ on the Lorentz space $L(p, q)$ into the linear space of all complex – valued measurable functions by

$$\mathcal{W}_{u,\mathrm{T}}(f)(x) = u\big(\mathrm{T}(x)\big) f\big(\mathrm{T}(x)\big), x \in X, f \in L(p, q).$$

If $\mathcal{W}$ is bounded with range in $L(p, q)$, then it is called a *weighted composition operator on* $L(p, q)$. if u $\equiv 1$,then $\mathcal{W} \equiv C_{\mathrm{T}}: f \rightarrow f \circ \mathrm{T}$ is called *composition* *operator* induced by T . If T is identity mapping , then $\mathcal{W} \equiv M_u : f \to u \cdot f$, a *multiplication operator* induced by u .The study of these operators on L_p -spaces has been made in **Chan1992, Jabbarzadeh and Pourreza 2003, Jabbarzadeh 2005 , Singh and Manhas 1993 and Takagi 1993**] and references there in .Composition and multiplication operators on the Lorentz spaces were studied in [**Kumar ,2005 , Arora , Datt and S. Verma 2006**] respectively .In this paper a characterization of the non – singular measurable transformations **T** from X into itself and complex –valued measurable function u on X inducing weighted composition operators is obtained on the Lorentz space $L(p,q)$, $0 < q,p < 1$.

2.Characterizations

In this section we introduce our main results .

Theorem 2.2. let $(X, \mathcal{A}, \mu)$ be a σ- finite measure space and $u: X \to \mathbb{C}$ be a measurable function . let $\mathrm{T}: X \to X$ be a non-singular measurable transformation such that the Radon-Nikodym derivative $f_{\mathrm{T}} = d(\mu \mathrm{T}^{-1})/d\mu$ is in $L_\infty(\mu)$.

Then $\mathcal{W}_{u,\mathrm{T}}: f \to u \circ \mathrm{T} \cdot f \circ \mathrm{T}$ is bounded on $L(p,q)$, $0 < q,p < 1$ if u$\in L_\infty(\mu)$.

Proof: Suppose b=$\|f_{\mathrm{T}}\|_\infty$, then for f in $L(p,q)$, the distribution function of $\mathcal{W}f$ satisfies , where $\mathcal{W}f = \mathcal{W}_{u,\mathrm{T}} = u \circ \mathrm{T} \cdot f \circ \mathrm{T}$, we have

$$(\mathcal{W}f)^{**}(\mathrm{t}) \le \|u\|_\infty f^{**}\left({}^{t}/_{b}\right) \quad \ldots.(1)$$ [**Arora , Datt and Verma , 2007**]

Then for $0 < q,p < 1$ we have

$$\|\mathcal{W}f\|_{pq}^q = \frac{q}{p}\int_0^\infty \left(t^{\frac{1}{p}}\left((\mathcal{W}f)^{**}(t)\right)\right)^q \frac{dt}{t}$$

Then by using (1) we have

$$\|\mathcal{W}f\|_{pq}^q \le \|u\|_\infty^q \frac{q}{p}\int_0^\infty \left(t^{\frac{1}{p}} f^{**}\left(\frac{t}{b}\right)\right)^q \frac{dt}{t}$$

$$= \|u\|_\infty^q \frac{q}{p}\int_0^{b\infty} \left((bt)^{\frac{1}{p}} f^{**}(t)\right)^q \frac{bdt}{bt}$$

$$\le \|u\|_\infty^q \; b^{\frac{q}{p}} \; \|f\|_{pq}^q$$

Thus

$$\|\mathcal{W}\|_{pq} \le b^{\frac{1}{p}} \|u\|_\infty$$

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